JOURNAL OF COMPUTATIONAL PHYSICS **124,** 286–300 (1996)
ARTICLE NO. 0060

An Approximate Linearized Riemann Solver for a Two-Fluid Model

I. TOUMI* AND A. KUMBARO†

*Commissariat à l'Energie Atomique, Centre d'Etudes de Saclay, DMT/SERMA, Gif-sur-Yvette, 91191, France; and
†Electricité De France, DER/PhR, 1, av. du Général de Gaulle, Clamart, 92140, France

Received March 31, 1993; revised July 28, 1995

An approximate linearized Riemann solver is presented for the numerical simulation of two-phase flows. This new solver is based on a linearization of nonconservative products and uses an extension of Roe's approximate Riemann solver. The scheme is applied to shock tube problem and to a standard test for two-fluid codes. © 1996 Academic Press, Inc.

1. INTRODUCTION

The linearized approximate Riemann solver of Roe [1] was proposed in 1981 for the numerical solution of the Euler equations governing the flow of an ideal gas. These last years, generalized Riemann solvers for the Euler equations with real gases have motivated many authors. Several extensions of Roe's linearization to an arbitrary equation of state have been proposed [2, 3].

A weak formulation of Roe's approximate Riemann solver, based on the choice of a path Φ in the states space, has been introduced in [4]. This weak formulation was applied in order to build a Roe-averaged matrix for a conservative system governing a homogeneous equilibrium two-phase flow.

We seek here to extend this formulation to a hyperbolic nonconservative system modelling a two-component two-phase flow. This flow consists in a mixture of water and steam, similar to those produced in pressurized water reactor cores. The model considered here is the isentropic equal pressure two-fluid model consisting in mass and momentum balance equations for each phase [5, 8]:

$$\begin{aligned} \partial_t(\alpha_k\rho_k) + \partial_x(\alpha_k\rho_k u_k) &= 0 \\ \partial_t(\alpha_k\rho_k u_k) + \partial_x(\alpha_k\rho_k u_k^2) + \alpha_k\partial_x p &= 0. \end{aligned} \tag{1.1}$$

The subscript k refers to the k-phase, p is the pressure, and α_k, ρ_k, u_k are respectively the volumetric average, the density, and the velocity of the k-phase. This system is naturally written in a nonconservative form due to the presence of nonconservative products $\alpha_k\partial_x p$ in the momentum equations. The main difficulty in the computation of two-phase flows, when separate mass and momentum equations are used for each phase, is the nonhyperbolic character of the governing equations [5, 11]. This leads to an ill-posed initial value problem which requires numerical damping terms to obtain stable results. The most common approach to solving these equations is a numerical method based on staggered grids and donor-cell differencing [6, 7]. This method, now almost universal in two-fluid codes like TRAC [9], RELAP [12], CATHARE [10], introduces a large amount of numerical diffusion. More accurate methods based on approximate Riemann solvers requires some modifications within the equations in order to obtain a hyperbolic system. In one of the first attempts at using a Godunov-type method for computing reactive multiphase flows, Toro [22] reinterpreted the equations so as to produce hyperbolic systems for each phase separately.

Following the leads of Drew and Lahey [15, 16], we introduce a virtual mass force term in the momentum equations. This interface exchange term which contains partial derivatives of the unknowns, makes the model hyperbolic. This enables us to apply numerical methods which make explicit use of the eigenvalues of the system.

However, there is another difficulty due to the nonconservative form of the system, since we know that the distribution theory is not suitable to define weak solutions in this case. Several approaches have been proposed to define weak solutions to nonconservative systems [18, 19]. In general, we need to add some information to the nonconservative system in order to obtain a complete definition of weak solutions. This information may be derived in different ways, for example, from a parabolic regularization of the system [17, 18]. From a numerical point of view, this nonuniqueness is reflected in the choice of the path Φ in the weak formulation of Roe's solver.

In the Section 2, we present the two-phase flow model whose numerical solutions are considered, and we look at the hyperbolicity of this model. Section 3 is devoted to the construction of an approximate linearized Riemann solver

0021-9991/96 $18.00

for nonconservative systems. The linearization of nonconservative products takes advantage of the fact that system (1.1) can formally be written in a conservative form. Then, following the method developed in [4], we construct a Roe averaged matrix using the canonical path for a parameter vector. We give in Section 5 some numerical results for shock tube problems and for the water faucet problem proposed by Ransom [13] as a benchmark for two-fluid codes.

2. TWO-PHASE FLOW MODEL

In this section we consider a one-dimensional two-fluid model dealing with an isentropic two-phase flow.

2.1. *Equations*

We begin with the first-order differential equations of mass and momentum conservation of two-fluid model which might describe isentropic two-component two-phase flow in a straight pipe:

$$\begin{aligned}
\partial_t(\alpha_v\rho_v) + \partial_x(\alpha_v\rho_v u_v) &= \Gamma_v \\
\partial_t(\alpha_l\rho_l) + \partial_x(\alpha_l\rho_l u_l) &= \Gamma_l \\
\partial_t(\alpha_v\rho_v u_v) + \partial_x(\alpha_v\rho_v u_v^2) + \alpha_v\partial_x p &= M_v \\
\partial_t(\alpha_l\rho_l u_l) + \partial_x(\alpha_l\rho_l u_l^2) + \alpha_l\partial_x p &= M_l
\end{aligned} \tag{2.1}$$

with

$$\alpha_v + \alpha_l = 1. \tag{2.2}$$

Here the subscript l refers to the liquid phase and v to the vapor phase; ρ_k, u_k, and α_k are the mass density, the velocity, and the void fraction of the k-phase, p is the pressure assumed to be equal in the two phases. To close the system the liquid phase is assumed to be incompressible, with constant mass density ρ_l, while the vapor mass density is given by the following state equation:

$$\rho_v = \rho_v(p). \tag{2.3}$$

The results, however, can be generalized to a compressible liquid phase. The source terms Γ_k and M_k include terms of interphase mass and momentum transfer, as well as losses via conduction and diffusion. We will assume that only the source term corresponding to the virtual mass force contains partial derivatives.

We have chosen the following formulation for the virtual mass force term M_{vm} at the right hand side of (2.1),

$$M_{vm} = -\alpha_v\alpha_l\rho c_{vm}\{\partial_t(u_v - u_l) + u_l\,\partial_x u_v - u_v\,\partial_x u_l\}, \tag{2.4}$$

where c_{vm} is the coefficient of the virtual mass and $\rho = \rho_v\alpha_v + \rho_l\alpha_l$ is the mixture density. Such formulation is used in the RELAP5 code [9] and is derived from the expression suggested by Drew [15],

$$\begin{aligned}
M_{vm} = \alpha_v\rho_l c_{vm}\{&\partial_t(u_v - u_l) + u_v\,\partial_x(u_v - u_l) \\
&+ (u_v - u_l)((\lambda - 2)\,\partial_x u_v + (1 - \lambda)\,\partial_x u_l)\},
\end{aligned} \tag{2.5}$$

where λ is a void fraction dependent parameter. The value of the coefficient of virtual mass term c_{vm}, is $\frac{1}{2}$ for noninteracting spheres and less than one-half for other shapes. Zuber [25] suggested the use of the following expression to account for the interaction effects

$$c_{vm} = \frac{1}{2}\left(\frac{1 + 2\alpha_d}{(1 - \alpha_d)}\right), \tag{2.6}$$

where α_d is the discontinuous phase fraction. We will define c_{vm} by a condition necessary to have a hyperbolic system.

Considering many interesting cases, the inclusion or neglect of the virtual mass force in the phasic momentum equations does not appreciably change the momentum results. However, the inclusion of this term with its temporal and spatial derivative terms has an effect on the hyperbolicity of the system. In general [26, 27] the computation efficiency of the solution scheme is improved by the inclusion of the virtual mass force term. The virtual mass force term formulation chosen is not the only one possible [21]. Another formulation will give similar results.

In what follows, excepted for the virtual mass force term, which contains partial derivatives, the other terms of mass and momentum transfer between phases are assumed to be absent. Otherwise they would appear as source terms. Then, the system considered in our study takes the following form:

$$\begin{aligned}
\partial_t(\alpha_v\rho_v) + \partial_x(\alpha_v\rho_v u_v) &= 0 \\
\partial_t(\alpha_l\rho_l) + \partial_x(\alpha_l\rho_l u_l) &= 0 \\
\partial_t(\alpha_v\rho_v u_v) + \partial_x(\alpha_v\rho_v u_v^2) + \alpha_v\,\partial_x p + M_{vm} &= 0 \\
\partial_t(\alpha_l\rho_l u_l) + \partial_x(\alpha_l\rho_l u_l^2) + \alpha_l\,\partial_x p - M_{vm} &= 0.
\end{aligned} \tag{2.7}$$

We define the concentration c:

$$c = \frac{\alpha_v \rho_v}{\alpha_v \rho_v + \alpha_l \rho_l}. \tag{2.8}$$

It will be useful to introduce also the pseudo-density ρ_p and the pseudo-concentration c_p:

$$\rho_p = \rho_v \alpha_l + \rho_l \alpha_v \tag{2.9}$$

$$c_p = \frac{\alpha_v \rho_l}{\alpha_v \rho_l + \alpha_l \rho_v}. \tag{2.10}$$

The system (2.7) is not written in a conservative form. However, we note that the mixture momentum ρu, given by

$$\rho u = \rho_v \alpha_v u_v + \rho_l \alpha_l u_l \tag{2.11}$$

satisfies the conservation equation

$$\partial_t(\rho u) + \partial_x(\alpha_v \rho_v u_v^2 + \alpha_l \rho_l u_l^2) + \partial_x p = 0. \tag{2.12}$$

The nonconservative terms in (2.7) arise from the splitting of this latter equation into two separate phase momentum equations. Introducing the mixture quantities ρ, ρu and the concentration variable c, we can write, for smooth solutions, system (2.7) in the nonconservative form

$$(I + M_\delta(\mathbf{u}))\, \partial_t \mathbf{u} + (A(\mathbf{u}) + A_\delta(\mathbf{u}))\, \partial_x \mathbf{u} = 0, \tag{2.13}$$

where the vector $\mathbf{u}$ and the matrix $A(\mathbf{u})$ are given by

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} \alpha_v \rho_v \\ \alpha_l \rho_l \\ \rho u \\ \alpha_l \rho_l u_l \end{bmatrix} \tag{2.14}$$

$$A(\mathbf{u}) = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ P_{u_1} - u_v^2 & P_{u_2} - u_l^2 & 2u_v & 2(u_l - u_v) \\ \alpha_l P_{u_1} & \alpha_l P_{u_2} - u_l^2 & 0 & 2u_l \end{bmatrix} \tag{2.15}$$

and the pressure derivatives with respect to the variables u_1 and u_2 are defined by

$$P_{u_1} = \left(\frac{\partial p}{\partial u_1}\right)_{u_2}, \quad P_{u_2} = \left(\frac{\partial p}{\partial u_2}\right)_{u_1}. \tag{2.16}$$

The matrices corresponding to the virtual mass force term are given by

$$A_\delta(\mathbf{u}) = \delta \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ (1-c)u_v u_l & -c u_v u_l & -(1-c)u_l & u \end{bmatrix} \tag{2.17}$$

and

$$M_\delta(\mathbf{u}) = \delta \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ (1-c)u_v & -c u_l & -(1-c) & 1 \end{bmatrix} \tag{2.18}$$

with

$$\delta = \frac{\rho^2}{\rho_v \rho_l} c_{vm}. \tag{2.19}$$

Assuming we overlook the terms arising from the virtual mass force in the system (2.13), we have the condensed form

$$\partial_t \mathbf{u} + A(\mathbf{u})\, \partial_x \mathbf{u} = 0. \tag{2.20}$$

This system is still nonconservative due to the term $\alpha_l\, \partial_x p$ in the liquid phase momentum equation. However, we prove the following.

PROPOSITION 2.1. *Let Ω be a set of physical states defined by*

$$\Omega = \{\mathbf{u}/(\rho > 0) \quad (c \in [0, 1])\}. \tag{2.21}$$

Let $\mathbf{u}$ be a continuous solution of system (2.20) *and let $\mathbf{v}$ be a vector valued function defined by*

$$\mathbf{v} = \begin{bmatrix} \alpha_v \rho_v \\ \alpha_l \rho_l \\ \rho u \\ u_l \end{bmatrix}. \tag{2.22}$$

The function $\mathbf{v}$ *is a solution of the conservative system*

$$\partial_t \mathbf{v} + \partial_x g(\mathbf{v}) = 0 \tag{2.23}$$

with the flux function defined by

$$g(\mathbf{v}) = \begin{bmatrix} \alpha_v \rho_v u_v \\ \alpha_l \rho_l u_l \\ \alpha_v \rho_v u_v^2 + \alpha_l \rho_l u_l^2 + p \\ \dfrac{u_l^2}{2} + \dfrac{p}{\rho_l} \end{bmatrix}. \tag{2.24}$$

Proof. Let $\mathbf{u}$ be a continuous solution of (2.20). We have to show that the liquid velocity u_l in our case satisfies the following conservative equation:

$$\partial_t u_l + \partial_x \frac{u_l^2}{2} + \frac{\partial_x p}{\rho_l} = 0. \tag{2.25}$$

We split the liquid phase momentum equation of system (2.20) into

$$u_l(\partial_t(\alpha_l \rho_l) + \partial_x(\alpha_l \rho_l u_l)) + \alpha_l \rho_l(\partial_t u_l + u_l\, \partial_x u_l) + \alpha_l\, \partial_x p = 0. \tag{2.26}$$

Then, using the liquid phase mass conservation equation and dividing the above equation by $\alpha_l \rho_l$ leads to Eq. (2.25). This settles the proof of Proposition 2.1.

Remark. Proposition 2.1 shows that the systems (2.23) and (2.20) are equivalent for smooth solutions. However, we are solving the nonconservative system, first, because this is the model used by many industrial codes [10, 12]. Second, this nonconservative two-phase flow model degenerates correctly to a one-phase flow model when the liquid void fraction tends to zero.

2.2. *Hyperbolicity of the System*

In order to study the hyperbolicity of the system (2.13) we are looking for the eigenvalues $\lambda_i(\mathbf{u})$ $(i = 1, 4)$ of this system. We recall that the system (2.13) is hyperbolic on Ω, if for any $\mathbf{u} \in \Omega$, all its eigenvalues are real. To determine these eigenvalues we must find the four roots of a polynomial of degree four $P_u(\lambda)$ given by

$$P_u(\lambda) = \det(A(\mathbf{u}) + A_\delta(\mathbf{u}) - \lambda(I + M_\delta(\mathbf{u}))). \tag{2.27}$$

The theory of algebraic equations shows us that it is possible to compute the four roots of this polynomial as algebraic functions of its coefficients. However, this would lead to very complicated computations which are not necessary to determine the behaviour of the roots $\lambda_i(\mathbf{u})$.

Actually, we prefer to use a perturbation method by introducing a small parameter ξ. We denote by c_m the speed of sound in the homogeneous two-phase mixture:

$$c_m = \left(\frac{\partial p}{\partial \rho}\right)_c^{1/2} = \left(\frac{\rho_v}{\alpha_v \rho}\right)^{1/2} c_v, \tag{2.28}$$

where c_v is the speed of sound in the vapor phase. The speed c_m is the usual speed of sound in the mixture obtained for the homogeneous (equal phase velocity) model. For a two-fluid model including unequal phase velocity and a virtual mass term, the natural speed of sound is given by

$$a_m = \left(\frac{\rho \rho_p + \rho^2 c_{vm}}{\rho_l \rho_p + \rho^2 c_{vm}}\right)^{1/2} c_m. \tag{2.29}$$

The speed of sound in stratified and homogeneous flows respectively correspond to the cases $c_{vm} = 0$ and $c_{vm} = \infty$. We assume that the relative velocity between the two phases is much lower than the speed of sound of the two-phase mixture which is the case in many physically interesting configurations, for example for steam and water. Then, the small parameter ξ will be given by the dimensionless relative velocity,

$$\xi = \frac{u_v - u_l}{a_m}, \tag{2.30}$$

and we will assume that $|\xi| \ll 1$. Using a perturbation method around $\xi = 0$ we prove the following.

PROPOSITION 2.2. *Let* ξ *be given by Eq.* (2.30) *and let* c_{vm}^0 *be equal to*

$$c_{vm}^0 = (4c(1 - c))^{1/2}. \tag{2.31}$$

We can find a positive number ξ_0 so that for any $\mathbf{u}$ *that lies in the set* Ω^* *defined by*

$$\Omega^* = \{\mathbf{u} \in \Omega : |\xi| \le \xi_0, c_{\mathrm{vm}} \ge c_{\mathrm{vm}}^0\}$$

the eigenvalues $\lambda_i(\mathbf{u})$, $i = 1, 4$, *are real and distinct. Moreover, these eigenvalues have the behaviour for* ξ *small enough*,

$$\begin{aligned}
\lambda_1 &= \frac{u_{\mathrm{v}} + u_{\mathrm{l}}}{2} - a_{\mathrm{m}} - \frac{1}{2}\left(\frac{(2c-1)\rho^2 c_{\mathrm{vm}}}{\rho_{\mathrm{l}}\rho_{\mathrm{v}} + \rho^2 c_{\mathrm{vm}}} + \frac{(2c_p - 1)\rho_p}{\rho_p + \rho c_{\mathrm{vm}}}\right) u_{\mathrm{r}} \\
&\quad + a_{\mathrm{m}} o\left(\frac{u_{\mathrm{r}}^2}{a_{\mathrm{m}}^2}\right) \\
\lambda_2 &= \frac{u_{\mathrm{v}} + u_{\mathrm{l}}}{2} + \frac{\rho_p}{2(\rho_p + \rho c_{\mathrm{vm}})} \\
&\quad \times \left(1 - 2c_p - \frac{\rho}{\rho_p}\sqrt{c_{\mathrm{vm}}^2 - (c_{\mathrm{vm}}^0)^2}\right) u_{\mathrm{r}} + a_{\mathrm{m}} o\left(\frac{u_{\mathrm{r}}^2}{a_{\mathrm{m}}^2}\right) \\
\lambda_3 &= \frac{u_{\mathrm{v}} + u_{\mathrm{l}}}{2} + \frac{\rho_p}{2(\rho_p + \rho c_{\mathrm{vm}})} \\
&\quad \times \left(1 - 2c_p + \frac{\rho}{\rho_p}\sqrt{c_{\mathrm{vm}}^2 - (c_{\mathrm{vm}}^0)^2}\right) u_{\mathrm{r}} + a_{\mathrm{m}} o\left(\frac{u_{\mathrm{r}}^2}{a_{\mathrm{m}}^2}\right) \\
\lambda_4 &= \frac{u_{\mathrm{v}} + u_{\mathrm{l}}}{2} + a_{\mathrm{m}} - \frac{1}{2}\left(\frac{(2c-1)\rho^2 c_{\mathrm{vm}}}{\rho_{\mathrm{l}}\rho_{\mathrm{v}} + \rho^2 c_{\mathrm{vm}}} + \frac{(2c_p - 1)\rho_p}{\rho_p + \rho c_{\mathrm{vm}}}\right) u_{\mathrm{r}} \\
&\quad + a_{\mathrm{m}} o\left(\frac{u_{\mathrm{r}}^2}{a_{\mathrm{m}}^2}\right),
\end{aligned} \tag{2.32}$$

where $u_{\mathrm{r}} = u_{\mathrm{v}} - u_{\mathrm{l}}$ *is the relative velocity between phases.*

Proposition 2.2 shows that the inclusion of virtual mass force term, with $c_{\mathrm{vm}} \ge c_{\mathrm{vm}}^0$, makes a well-posed model.

Proof. It is convenient to introduce the dimensionless variable

$$z = \frac{1}{a_{\mathrm{m}}}\left(\lambda - \frac{u_{\mathrm{v}} + u_{\mathrm{l}}}{2}\right). \tag{2.33}$$

We can then rewrite [20] the polynomial $P_u(\lambda)$ as

$$P_u(z;\xi) = p_0(z) + p_1(z)\xi + p_2(z)\frac{\xi^2}{2} + q(z;\xi)\xi^2 \tag{2.34}$$

with

$$|q(z;\xi)| \le (1 + |z|)^s \varphi(\xi) \quad \text{if } s \in N \text{ and } \exists \lim_{\xi \to 0} \varphi(\xi) = 0. \tag{2.35}$$

We look for the roots of the polynomial $P_u(z, \xi)$ in a neighbourhood of a root z_0 of the polynomial $p_0(z)$. We distinguish two cases, whether z_0 is a simple root or a double root [23, 24].

First case. z_0 is a simple root, $z_0 = \pm 1$. Then, the first-order approximation, defined for $\xi \ne 0$ close enough to zero, is given by

$$z(\xi) = z_0 + z_1\xi + o(\xi^2) \tag{2.36}$$

with

$$z_1 = -\frac{p_1(z_0)}{p_0'(z_0)}. \tag{2.37}$$

Second case. z_0 is a double root of $p_0(z)$, $z_0 = 0$, and $p_1(z_0) = 0$. Then, at first order in ξ, the double root splits into two simple roots, defined for $\xi \ne 0$ close enough to zero and given by

$$z^{\pm}(\xi) = z_0 + z_1^{\pm}\xi + o(\xi^2), \tag{2.38}$$

where $z_1^{\pm}$ are the two roots of the equation

$$p_0''(z_0)z_1^2 + 2p_1'(z_0)z_1 + p_2(z_0) = 0. \tag{2.39}$$

Consequently, the roots remain real at first order and the system is hyperbolic if the condition $[(p_1'(0))^2 - p_2(0)p_0''(0)] > 0$ is satisfied. This condition is equivalent to

$$c_{\mathrm{vm}} \ge c_{\mathrm{vm}}^0 = (4c(1-c))^{1/2}. \tag{2.40}$$

Finally, using (2.30) and (2.33)–(2.40), we obtain the first-order approximation in ξ of the eigenvalues given by the expressions (2.32).

3. AN APPROXIMATE RIEMANN SOLVER FOR NONCONSERVATIVE SYSTEMS

In Godunov's method [23], the approximate solution $\mathbf{u}^{n+1}$, at time $(n + 1)\,\Delta t$, is obtained by solving Riemann problem at cell interfaces

$$\begin{aligned}
\mathbf{u}_i^{n+1} &= \mathbf{u}_i^n + \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_i} u_e\left(\frac{x - x_{i-1/2}}{\Delta t}, \mathbf{u}_{i-1}^n, \mathbf{u}_i^n\right) dx \\
&\quad + \frac{1}{\Delta x}\int_{x_i}^{x_{i+1/2}} u_e\left(\frac{x - x_{i+1/2}}{\Delta t}, \mathbf{u}_i^n, \mathbf{u}_{i+1}^n\right) dx,
\end{aligned} \tag{3.0}$$

where $u_e((x - x_{i-1/2})/\Delta t, \mathbf{u}_{i-1}^n, \mathbf{u}_i^n)$ is the exact solution of the Riemann problem

$$\begin{gathered}\partial_t \mathbf{u} + \partial_x f(\mathbf{u}) = 0 \\ \mathbf{u}(x, 0) = \mathbf{u}_{i-1}^n \ (x < x_{i-1/2}), \quad \mathbf{u}(x, 0) = \mathbf{u}_i^n \ (x > x_{i-1/2}).\end{gathered}$$

The solution of Riemann problem being unknown for our nonconservative model, we cannot construct a Godunov type numerical scheme [24]. Then, we suggest the use of an approximate Riemann solver.

3.1. *Extension of Roe's Scheme*

To solve the nonlinear Riemann problem for hyperbolic systems of conservation laws,

$$\begin{gathered}\partial_t \mathbf{u} + \partial_x f(\mathbf{u}) = 0 \\ \mathbf{u}(x, 0) = \mathbf{u}_{\mathrm{L}} \ (x < 0), \quad \mathbf{u}(x, 0) = \mathbf{u}_{\mathrm{R}} \ (x > 0)\end{gathered} \tag{3.1}$$

Roe [1] introduces a local linearization

$$\partial_t \mathbf{u} + A(\mathbf{u}_{\mathrm{L}}, \mathbf{u}_{\mathrm{R}}) \, \partial_x \mathbf{u} = 0, \tag{3.2}$$

where $A(\mathbf{u}_{\mathrm{L}}, \mathbf{u}_{\mathrm{R}})$ is some average Jacobian matrix, known as a Roe-averaged matrix. The matrix $A(\mathbf{u}_{\mathrm{L}}, \mathbf{u}_{\mathrm{R}})$ is constructed to have the following property:

$$f(\mathbf{u}_{\mathrm{R}}) - f(\mathbf{u}_{\mathrm{L}}) = A(\mathbf{u}_{\mathrm{L}}, \mathbf{u}_{\mathrm{R}})(\mathbf{u}_{\mathrm{R}} - \mathbf{u}_{\mathrm{L}}). \tag{3.3}$$

This property ensures that shocks of linear system (3.2) are shocks of the nonlinear system (3.1), satisfying the Rankine–Hugoniot condition:

$$\sigma(\mathbf{u}_{\mathrm{R}} - \mathbf{u}_{\mathrm{L}}) = f(\mathbf{u}_{\mathrm{R}}) - f(\mathbf{u}_{\mathrm{L}}). \tag{3.4}$$

Such a matrix $A(\mathbf{u}_{\mathrm{L}}, \mathbf{u}_{\mathrm{R}})$ was first constructed by Roe for Euler equations with perfect gases [1], and several extensions to real gases have been proposed (see [4] and the references therein). However, Roe's method does not apply to a nonconservative system,

$$\partial_t \mathbf{u} + A(\mathbf{u}) \, \partial_x \mathbf{u} = 0, \tag{3.5}$$

for which the matrix $A(\mathbf{u})$ is not the derivative of a flux function $f(\mathbf{u})$. Specifically, we need to define, for nonconservative systems, a condition equivalent to condition (3.3), based upon a generalized Rankine–Hugoniot condition.

Thus, we use a weak formulation of Roe's approximate Riemann solver introduced in [4]. In this formulation, we consider approximate solutions to the nonlinear Riemann problem which are exact solutions to the linear problem

$$\begin{gathered}\partial_t \mathbf{u} + A(\mathbf{u}_{\mathrm{L}}, \mathbf{u}_{\mathrm{R}})_\Phi \, \partial_x \mathbf{u} = 0 \\ \mathbf{u}(x, 0) = \mathbf{u}_{\mathrm{L}} \ (x < 0), \quad \mathbf{u}(x, 0) = \mathbf{u}_{\mathrm{R}} \ (x > 0),\end{gathered} \tag{3.6}$$

where $A(\mathbf{u}_{\mathrm{L}}, \mathbf{u}_{\mathrm{R}})_\Phi$ is a matrix depending on the data $(\mathbf{u}_{\mathrm{L}}, \mathbf{u}_{\mathrm{R}})$ and on a smooth path $\Phi(s, \mathbf{u}_{\mathrm{L}}, \mathbf{u}_{\mathrm{R}})$ linking the states $\mathbf{u}_{\mathrm{L}}$ and $\mathbf{u}_{\mathrm{R}}$. The matrix $A(\mathbf{u}_{\mathrm{L}}, \mathbf{u}_{\mathrm{R}})_\Phi$ must satisfy the following property:

$$\int_0^1 A(\Phi(s; \mathbf{u}_{\mathrm{L}}, \mathbf{u}_{\mathrm{R}})) \frac{\partial \Phi}{\partial s}(s; \mathbf{u}_{\mathrm{L}}, \mathbf{u}_{\mathrm{R}}) \, ds = A(\mathbf{u}_{\mathrm{L}}, \mathbf{u}_{\mathrm{R}})_\Phi (\mathbf{u}_{\mathrm{L}} - \mathbf{u}_{\mathrm{R}}). \tag{3.7}$$

This condition, which is a generalization of Roe's condition (3.3), shows that a shock $(\mathbf{u}_{\mathrm{L}}, \mathbf{u}_{\mathrm{R}})$ with speed σ, satisfies the generalized Rankine–Hugoniot condition given in [18]

$$\int_0^1 (A(\Phi(s; \mathbf{u}_{\mathrm{L}}, \mathbf{u}_{\mathrm{R}})) - \sigma I) \frac{\partial \Phi}{\partial s}(s; \mathbf{u}_{\mathrm{L}}, \mathbf{u}_{\mathrm{R}}) \, ds = 0. \tag{3.8}$$

We refer the reader to [4] for more details on this formulation of Roe's approximate solver and its application to conservative two-phase flow models.

We note that for a conservative system the right-hand side of (3.7) is independent of the path Φ:

$$\int_0^1 A(\Phi(s; \mathbf{u}_{\mathrm{L}}, \mathbf{u}_{\mathrm{R}})) \frac{\partial \Phi}{\partial s}(s; \mathbf{u}_{\mathrm{L}}, \mathbf{u}_{\mathrm{R}}) \, ds = f(\mathbf{u}_{\mathrm{R}}) - f(\mathbf{u}_{\mathrm{L}}).$$

Thus, (3.7) coincides exactly with Roe's condition (3.3). A shock wave solution of the linearized system satisfies the Rankine–Hugoniot condition for the nonlinear conservative system (3.1) and is independent of the path Φ. In this case, the path Φ is only useful to linearize the Jacobian matrix $A(\mathbf{u})$ to obtain $A(\mathbf{u}_{\mathrm{L}}, \mathbf{u}_{\mathrm{R}})_\Phi$.

On the other hand, we remark that the above definition (3.7) does not require a flux function $f(\mathbf{u})$. Thereby, this weak formulation allows us to construct approximate Riemann solvers for hyperbolic nonconservative systems. However, for such systems, the choice of the path Φ will be important because both the exact solution and the approximate solver are strongly dependent on the path. In order to clarify the choice of the path Φ for nonconservative systems, we propose to separate the path contributions on two above problems:

—the definition of shock wave solutions for nonconservative systems which needs a path with a physical meaning

—the linearization of the nonlinear matrix $A(\mathbf{u})$ which does not need a physical path.

3.2. *Nonconservative System Case*

Concerning the first point, some authors [17, 18], suggest an approach motivated by physical considerations. Let us mention the work of Sainsaulieu [17], which introduces travelling waves $\mathbf{u}(x,t) = \hat{\mathbf{u}}(x - \sigma t)$, σ is the shock propagation velocity, solutions of the second-order convection–diffusion system

$$\partial_t \mathbf{u} + A(\mathbf{u})\,\partial_x \mathbf{u} - \partial_x(D(\mathbf{u})\,\partial_x \mathbf{u}) = 0, \tag{3.9}$$

where $D(\mathbf{u})$ is a diffusion tensor. If $\mathbf{u}(x, t)$ is a travelling wave solution of (3.9), the vector valued function $\mathbf{v}_\varepsilon(x) = \hat{\mathbf{u}}(x/\varepsilon)$ is a solution of the differential system:

$$-\sigma \mathbf{v}_\varepsilon' + A(\mathbf{v}_\varepsilon)\mathbf{v}_\varepsilon' - \varepsilon(D(\mathbf{v}_\varepsilon)\mathbf{v}_\varepsilon')' = 0.$$

Then, we can define a shock wave solution of the system (3.5) as the limit $\lim_{\varepsilon\to 0} \mathbf{v}_\varepsilon(x)$, when the diffusion is neglected, of travelling wave solution of (3.9). When ε tends to zero, $\mathbf{v}_\varepsilon(x)$ converges to the following discontinuous function:

$$\mathbf{v}_0(x) = \begin{cases} \hat{\mathbf{u}}(-\infty) & \text{if } x < 0 \\ \hat{\mathbf{u}}(+\infty) & \text{if } x > 0. \end{cases}$$

Turning back to our approximate Riemann solver and using a ''viscous'' path,

$$\Phi(s; \mathbf{u}_\mathrm{L}, \mathbf{u}_\mathrm{R}), \quad s \in [0,1], \tag{3.10}$$

which is the viscous profile linking the states $\mathbf{u}_\mathrm{L} = \hat{\mathbf{u}}(-\infty)$ and $\mathbf{u}_\mathrm{R} = \hat{\mathbf{u}}(+\infty)$, solution of (3.9), Eq. (3.7) becomes

$$A(\mathbf{u}_\mathrm{L}, \mathbf{u}_\mathrm{R})_\Phi(\mathbf{u}_\mathrm{R} - \mathbf{u}_\mathrm{L}) = h(\mathbf{u}_\mathrm{R}) - h(\mathbf{u}_\mathrm{L}) + \tilde{\alpha}_1(\mathbf{p}_\mathrm{R} - \mathbf{p}_\mathrm{L}). \tag{3.11}$$

The flux function $h(\mathbf{u})$ and the vector $\mathbf{p}$ are defined by

$$h(\mathbf{u}) = \begin{bmatrix} \alpha_\mathrm{v}\rho_\mathrm{v}u_\mathrm{v} \\ (1-\alpha_\mathrm{l})\rho_\mathrm{l}u_\mathrm{l} \\ \alpha\rho_\mathrm{v}u_\mathrm{v}^2 + (1-\alpha_\mathrm{l})\rho_\mathrm{l}u_\mathrm{l}^2 + p \\ (1-\alpha_\mathrm{l})\rho_\mathrm{l}u_\mathrm{l}^2 \end{bmatrix},$$

$$\mathbf{p} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ p \end{bmatrix}$$

and the average void fraction $\tilde{\alpha}_\mathrm{l}$ is given by

$$\tilde{\alpha}_1 = \frac{1}{p_\mathrm{R} - p_\mathrm{L}} \int_0^1 \alpha_1 \frac{\partial p}{\partial \mathbf{u}} \frac{\partial \Phi}{\partial s}(s, \mathbf{u}_\mathrm{L}, \mathbf{u}_\mathrm{R})\, ds. \tag{3.12}$$

The integral in (3.12) is dependent on the path Φ and may be difficult to compute, since the definition of the path requires to solve the differential system (3.9). As this ''viscous'' path will not be used for the linearization of the matrix $A(\mathbf{u})$, it would be preferable to calculate by other means, the average void fraction $\tilde{\alpha}_\mathrm{l}$. The following proposition gives the average $\tilde{\alpha}_\mathrm{l}$ for which the nonconservative system (2.20) and the conservative system (2.23) have the same shock waves.

PROPOSITION 3.2. *Let $\tilde{\alpha}_l$ be defined by*

$$\tilde{\alpha}_\mathrm{l} = 2\,\frac{\alpha_\mathrm{l}^\mathrm{L}\alpha_\mathrm{l}^\mathrm{R}}{\alpha_\mathrm{l}^\mathrm{L} + \alpha_\mathrm{l}^\mathrm{R}}; \tag{3.13}$$

then the nonconservative system (2.20) *and the conservative system* (2.23) *have the same Rankine–Hugoniot conditions.*

Proof. Since the first three equations of systems (2.20) and (2.23) are the same, we have to compare the Rankine–Hugoniot conditions for the fourth equation. From Eqs. (3.8) and (3.11), we obtain for the nonconservative system

$$[\alpha_\mathrm{l}\rho_\mathrm{l}u_\mathrm{l}^2] + \tilde{\alpha}_\mathrm{l}[p] = \sigma[\alpha_\mathrm{l}\rho_\mathrm{l}u_\mathrm{l}], \tag{3.14}$$

where $[\cdot] = (\cdot)_\mathrm{R} - (\cdot)_\mathrm{L}$. Using the mass conservation for the liquid phase, we get

$$M = (\alpha_\mathrm{l}\rho_\mathrm{l}(\sigma - u_\mathrm{l}))_\mathrm{L} = (\alpha_\mathrm{l}\rho_\mathrm{l}(\sigma - u_\mathrm{l}))_\mathrm{R} \tag{3.15}$$

and

$$\tilde{\alpha}_\mathrm{l}[p] = M[u_\mathrm{l}]. \tag{3.16}$$

On the other hand, the Rankine–Hugoniot condition for the conservative system (2.23) gives

$$\left(\frac{u_\mathrm{l}^2}{2} + \frac{p}{\rho_\mathrm{l}}\right) = \sigma[u_\mathrm{l}]. \tag{3.17}$$

From Eqs. (3.16) and (3.17) we deduce

$$\frac{M}{\rho_\mathrm{l}\tilde{\alpha}_\mathrm{l}} = \sigma - \frac{1}{2}(u_\mathrm{l}^\mathrm{L} + u_\mathrm{l}^\mathrm{R}). \tag{3.18}$$

Finally, using (3.18) and (3.15), we obtain the average void fraction given by (3.13).

3.3. *Conservative System Case*

Using the average $\tilde{\alpha}_l$ suggested by Proposition 3.1, we have to solve now the Riemann problem for the conservative system

$$\partial_t \mathbf{u} + \partial_x \tilde{h}(u, \tilde{\alpha}_l) = 0 \tag{3.19}$$

with

$$\tilde{h}(\mathbf{u}, \tilde{\alpha}_l) = \begin{bmatrix} \alpha_v \rho_v u_v \\ \alpha_l \rho_l u_l \\ \alpha_v \rho_v u_v^2 + \alpha_l \rho_l u_l^2 + p \\ \alpha_l \rho_l u_l^2 + \tilde{\alpha}_l p \end{bmatrix}. \tag{3.20}$$

Thus, the path Φ will only have an effect upon the linearization of the nonlinear Jacobian matrix

$$A(\mathbf{u}, \tilde{\alpha}_l) = \frac{\partial \tilde{h}(\mathbf{u}, \tilde{\alpha}_l)}{\partial \mathbf{u}}.$$

To construct the Roe-type matrix $A(\mathbf{u}_L, \mathbf{u}_R)_\Phi$, we follow the method introduced in [4]. The main feature is the choice of the canonical path for a parameter vector $\mathbf{w}$,

$$\Phi(s; \mathbf{u}_L, \mathbf{u}_R) = \varphi_0(\mathbf{w}_L + s(\mathbf{w}_R - \mathbf{w}_L)),$$

where φ_0 is a smooth function such that $\varphi_0(\mathbf{w}_L) = \mathbf{u}_L$, $\varphi_0(\mathbf{w}_R) = \mathbf{u}_R$, and $A_0(\mathbf{w}) = \partial\varphi_0/\partial\mathbf{w}$ is a regular matrix for every state $\mathbf{w}$. Using this path, we define Roe's matrix by

$$A(\mathbf{u}_L, \mathbf{u}_R)_\Phi = C(\mathbf{u}_L, \mathbf{u}_R)_\Phi B(\mathbf{u}_L, \mathbf{u}_R)_\Phi^{-1} \tag{3.20}$$

with

$$B(\mathbf{u}_L, \mathbf{u}_R)_\Phi = \int_0^1 A_0(\mathbf{w}_L + s(\mathbf{w}_R - \mathbf{w}_L))\, ds$$

$$C(\mathbf{u}_L, \mathbf{u}_R)_\Phi = \int_0^1 A(\varphi_0(\mathbf{w}_L + s(\mathbf{w}_R - \mathbf{w}_L))) \times A_0(\mathbf{w}_L + s(\mathbf{w}_R - \mathbf{w}_L))\, ds.$$

The choice of the canonical path is motivated by the results obtained for the Euler equations and the homogeneous two-phase flow model.

4. APPLICATION TO THE TWO-FLUID MODEL

In this section, using the above weak formulation, we build an approximate Riemann solver for the two-fluid model (2.7). First, we construct the Roe-averaged matrix for the system (2.20) which does not include the virtual mass force term. Then, we will extend the method for the complete system.

4.1. *Two-Fluid Model without Mass Term*

We apply the method presented in Section 3 with the parameter vector chosen as

$$\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} \sqrt{\alpha_v \rho_v} \\ \sqrt{\alpha_l \rho_l} \\ \sqrt{\alpha_v \rho_v} u_v \\ \sqrt{\alpha_l \rho_l} u_l \end{bmatrix} \tag{4.1}$$

and $\varphi_0(\mathbf{w})$ given by the expression

$$\varphi_0(\mathbf{w}) = \begin{bmatrix} w_1^2 \\ w_2^2 \\ w_1 w_3 + w_2 w_4 \\ w_2 w_4 \end{bmatrix}. \tag{4.2}$$

Straightforward computations yield

$$B(\mathbf{u}_L, \mathbf{u}_R)_\Phi = \begin{bmatrix} 2\overline{w}_1 & 0 & 0 & 0 \\ 0 & 2\overline{w}_2 & 0 & 0 \\ \overline{w}_3 & \overline{w}_4 & \overline{w}_1 & \overline{w}_2 \\ 0 & \overline{w}_4 & 0 & \overline{w}_2 \end{bmatrix} \tag{4.3}$$

and

$$C(\mathbf{u}_L, \mathbf{u}_R)_\Phi = \begin{bmatrix} \overline{w}_3 & 0 & \overline{w}_1 & 0 \\ 0 & \overline{w}_4 & 0 & \overline{w}_2 \\ \tilde{P}_{w_1} & \tilde{P}_{w_2} & 2\overline{w}_3 & 2\overline{w}_4 \\ \tilde{\alpha}_l \tilde{P}_{w_1} & \tilde{\alpha}_l \tilde{P}_{w_2} & 0 & 2\overline{w}_4 \end{bmatrix}, \tag{4.4}$$

where $\overline{w}_i$ denotes the arithmetic mean of w_i and $\tilde{P}_{w_i}$ is an average of the pressure derivative $(\partial p/\partial w_i)_{w_j}$ given by

$$\tilde{P}_{w_i} = \int_0^1 \frac{\partial p}{\partial w_i}(\mathbf{w}_L + s(\mathbf{w}_R - \mathbf{w}_L))\, ds. \tag{4.5}$$

Finally, using Eq. (3.20), we find the Roe-averaged matrix for the system without the virtual mass term

$$A(\mathbf{u}_L, \mathbf{u}_R)_\Phi = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ \tilde{P}_1 - \tilde{u}_v^2 & \tilde{P}_2 - \tilde{u}_l^2 & 2\tilde{u}_v & \tilde{u}_l - \tilde{u}_v \\ \tilde{\alpha}_l P_1 & \tilde{\alpha}_l \tilde{P}_2 - \tilde{u}_l^2 & 0 & 2\tilde{u}_l \end{bmatrix}, \quad (4.6)$$

where $\tilde{P}_i$ is an approximation of the pressure derivative $(\partial p/\partial u_i)_{u_j}$ given by

$$\tilde{P}_i = \frac{\tilde{P}_{w_i}}{2\overline{w}_i}$$

and $\tilde{u}_k$ is a Roe-averaged velocity given by

$$\tilde{u}_k = \frac{\sqrt{(\rho_k \alpha_k)^R} u_k^R + \sqrt{(\rho_k \alpha_k)^L} u_k^L}{\sqrt{(\rho_k \alpha_k)^R} + \sqrt{(\rho_k \alpha_k)^L}}. \quad (4.7)$$

4.2. *Two-Fluid Model with Virtual Mass Term*

We build now the approximate Riemann solver for the complete system (2.7). The previous section shows that the canonical path for Roe's parameter vector $\mathbf{w}$, leads to a linearized Jacobian matrix which is equal to the exact Jacobian evaluated at some average state $\tilde{\mathbf{u}}$ defined by $\tilde{u}_k$, $\tilde{P}_i$, and $\tilde{\alpha}_l$,

$$A(\mathbf{u}_L, \mathbf{u}_R)_\Phi = A(\tilde{\mathbf{u}}). \quad (4.8)$$

As for the exact Jacobian matrix $A(\mathbf{u})$, the linearized matrix $A(\mathbf{u}_L, \mathbf{u}_R)_\Phi$ has generally two complex eigenvalues. We expect that the inclusion of the virtual mass term leads to a hyperbolic linearized system, without sensibly changing the momentum results. We have to linearize this new nonconservative system, but there is no equivalent conservative system that would inspire the linearization of our system. Thus, we propose to linearize the matrices $A_\delta(\mathbf{u})$ and $M_\delta(\mathbf{u})$ using the same average state $\tilde{\mathbf{u}}$. Then, the linearization of system (2.13) is given by

$$(I + M_\delta(\tilde{\mathbf{u}}))\partial_t \mathbf{u} + (A(\mathbf{u}_L, \mathbf{u}_R)_\Phi + A_\delta(\tilde{\mathbf{u}}))\partial_x \mathbf{u} = 0. \quad (4.9)$$

A routine calculation shows that the system (4.9) may be written under the form

$$\partial_t \mathbf{u} + A(\mathbf{u}_L, \mathbf{u}_R)_\Phi^{vm} \partial_x \mathbf{u} = 0, \quad (4.10)$$

where the Roe-averaged matrix is given as a sum of two averaged matrices

$$A(\mathbf{u}_L, \mathbf{u}_R)_\Phi^{vm} = A(\mathbf{u}_L, \mathbf{u}_R)_\Phi + \tilde{\delta} A_\delta(\mathbf{u}_L, \mathbf{u}_R)_\Phi. \quad (4.11)$$

The first matrix is the Roe-averaged matrix for the system without the virtual mass term and the second is the linearized matrix corresponding to the virtual mass term.

5. NUMERICAL RESULTS

Once the Roe-averaged matrix has been constructed, the linear Riemann problem is relatively easy to solve by using its eigenvalues $\lambda_{i-1/2}^k$ and its eigenvectors $R_{i-1/2}^k$. If we decompose

$$\mathbf{u}_i^n - \mathbf{u}_{i-1}^n = \sum_k \beta_{i-1/2}^k R_{i-1/2}^k \quad (5.1)$$

then, the exact solution of the linear Riemann problem (3.6) and the approximate solution of (3.5) are given by

$$\mathbf{u}_a(\xi, \mathbf{u}_{i-1}^n, \mathbf{u}_i^n) = \mathbf{u}_{i-1}^n + \sum_{\lambda_{i-1/2}^k < \xi} \beta_{i-1/2}^k R_{i-1/2}^k, \quad (5.2)$$

where the sum is over all the eigenvalues for which $\lambda_{i-1/2}^k < \xi$. Equivalently,

$$\mathbf{u}_a(\xi, \mathbf{u}_{i-1}^n, \mathbf{u}_i^n) = \mathbf{u}_i^n - \sum_{\lambda_{i-1/2}^k > \xi} \beta_{i-1/2}^k R_{i-1/2}^k. \quad (5.3)$$

Substituting this approximate solution into (3.0) and setting $\xi = (x - x_{i-1/2})/\Delta t$ in the first integral and $\xi = (x - x_{i+1/2}/\Delta t$ in the second one leads to

$$\mathbf{u}_i^{n+1} = \mathbf{u}_i^n + \frac{\Delta t}{\Delta x} \int_0^{\Delta x/2\Delta t} \mathbf{u}_e(\xi, \mathbf{u}_{i-1}^n, \mathbf{u}_i^n)\, d\xi + \frac{\Delta t}{\Delta x} \int_{-\Delta x/2\Delta t}^0 \mathbf{u}_e(\xi, \mathbf{u}_i^n, \mathbf{u}_{i+1/2}^n)\, d\xi. \quad (5.4)$$

The expression of the numerical scheme follows from (5.2) and (5.3),

$$\mathbf{u}_i^{n+1} = \mathbf{u}_i^n - \frac{\Delta t}{\Delta x}(F^-(\mathbf{u}_i^n, \mathbf{u}_{i+1}^n) + F^+(\mathbf{u}_{i-1}^n, \mathbf{u}_i^n)), \quad (5.5)$$

written in a nonconservative form

$$F^\pm(\mathbf{u}_{i-1}^n, \mathbf{u}_i^n) = A^\pm(\mathbf{u}_{i-1}^n, \mathbf{u}_i^n)_\Phi(\mathbf{u}_i^n - \mathbf{u}_{i-1}^n). \quad (5.6)$$

The positive and negative part of the Roe-averaged matrix are given by

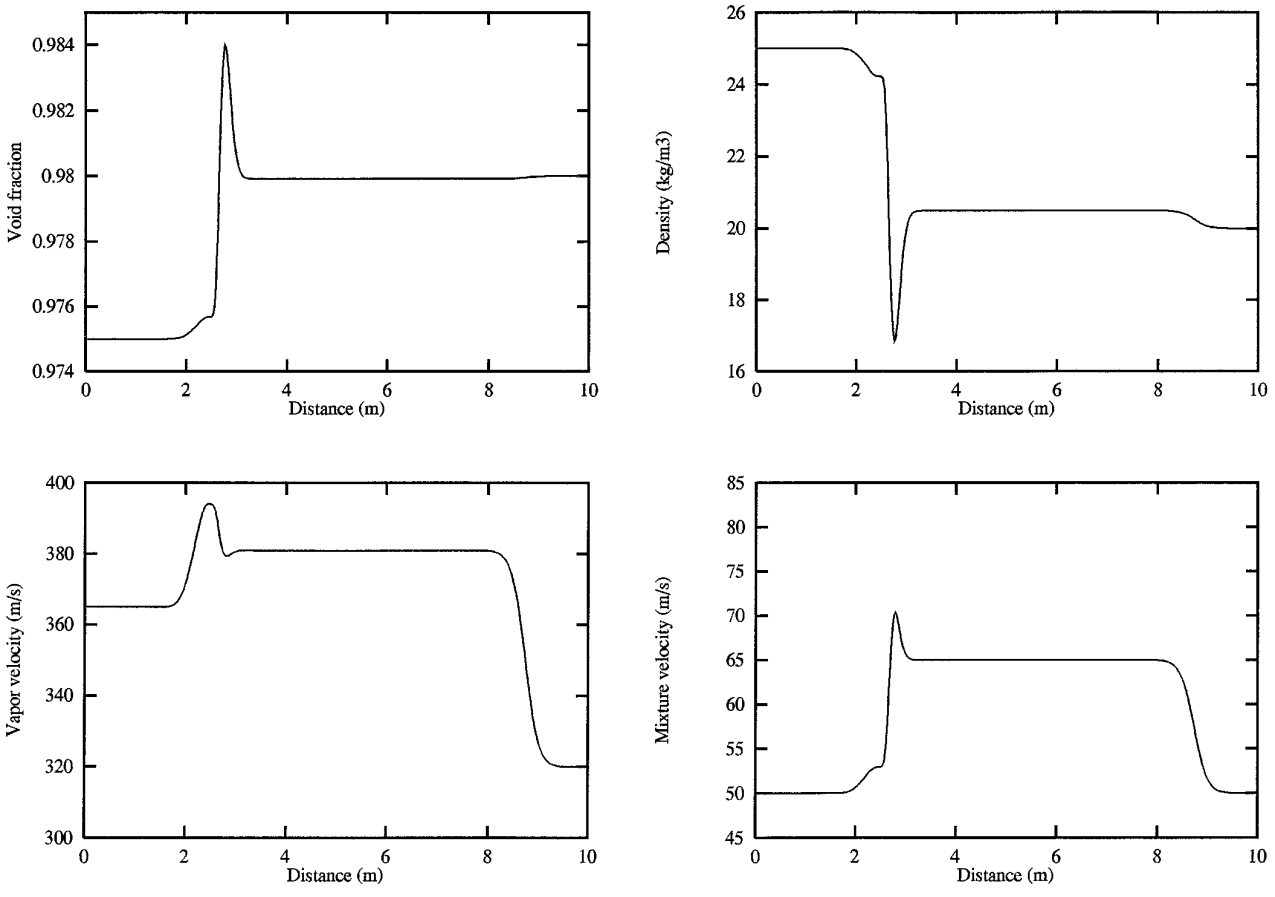


FIG. 1. Riemann problem with $c_{\text{vm}} = 2$.

$$A^{\pm}(\mathbf{u}_{i-1}^n, \mathbf{u}_i^n)_{\Phi} = R_{i-1/2}\Lambda_{i-1/2}^{\pm}R_{i-1/2}^{-1}, \tag{5.7}$$

where $R_{i-1/2}$ is the matrix containing the eigenvectors of the Roe-averaged matrix, and $\Lambda_{i-1/2}^{\pm}$ is the diagonal matrix containing the positive and the negative part of the eigenvalues

$$\Lambda_{i-1/2}^{\pm} = \text{diag}(\lambda_{i-1/2}^{1\pm}, ..., \lambda_{i-1/2}^{m\pm}) \tag{5.8}$$

with

$$\lambda_{i-1/2}^{k-} = \min(0, \lambda_{i-1/2}^k), \quad \lambda_{i-1/2}^{k+} = \max(0, \lambda_{i-1/2}^k). \tag{5.9}$$

We present now some test problems and show the numerical results obtained, using the approximate Riemann solver built in Section 4 and the first-order numerical scheme (5.5).

PROBLEM 1 (Shock tube problem). This problem consists in a Riemann problem for the system (2.7), where the left and right states are given by

State $\mathbf{u}_{\text{L}}$. $\rho_{\text{L}} = 25$ kgm^{-3}, $c_{\text{L}} = 0.1$, $u_{\text{L}} = 50$ ms^{-1}, $u_{\text{l}_{\text{L}}} = 15$ ms^{-1},

State $\mathbf{u}_{\text{R}}$. $\rho_{\text{R}} = 20$ kgm^{-3}, $c_{\text{R}} = 0.1$, $u_{\text{R}} = 50$ ms^{-1}, $u_{\text{l}_{\text{R}}} = 20$ ms^{-1}.

The vapor phase is assumed to be an ideal isentropic gas. The computations have been done with 300 nodes and using a virtual mass coefficient equal to 2 and 50. The former value is closer to the physical value of this coefficient. Figures 1 and 2 give some flow characteristics in each case. The solution is composed by five constant states separated by rarefaction waves or shock waves. The propagation velocities of the second and third waves being close to each other for small values of the virtual mass coefficient, these waves are not well separated in Fig. 1.

PROBLEM 2 (More on the shock tube problem). This problem consists in a Riemann problem for the system (2.7), where the left and right states are given by

State $\mathbf{u}_{\text{L}}$. $\rho_{\text{L}} = 1.9$ kgm^{-3}, $c_{\text{L}} = 0.3$, $u_{\text{L}} = 10$ ms^{-1}, $u_{\text{l}_{\text{L}}} = 6$ ms^{-1},

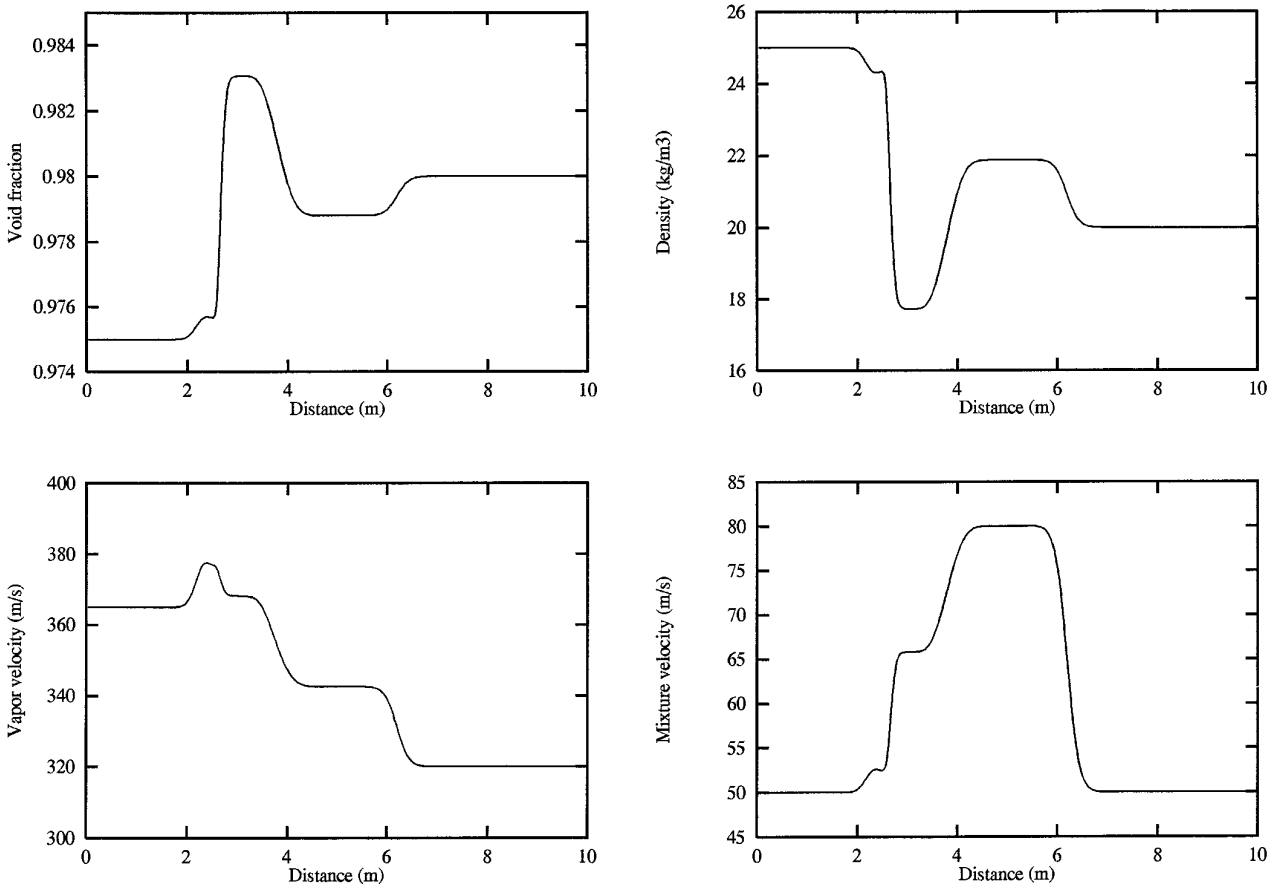


FIG. 2. Riemann problem with $c_{vm} = 50$.

State $\mathbf{u}_R$. $\rho_R = 30\ \text{kgm}^{-3}$, $c_R = 0.1$, $u_R = 10\ \text{ms}^{-1}$, $u_{l_R} = 6\ \text{ms}^{-1}$.

The computations have been done with 300 nodes and using a virtual mass coefficient equal to 100. In Section 4 we proved that taking a void fraction average given by

$$\tilde{\alpha}_l = 2\,\frac{\alpha_l^L \alpha_l^R}{\alpha_l^L + \alpha_l^R}, \tag{5.6}$$

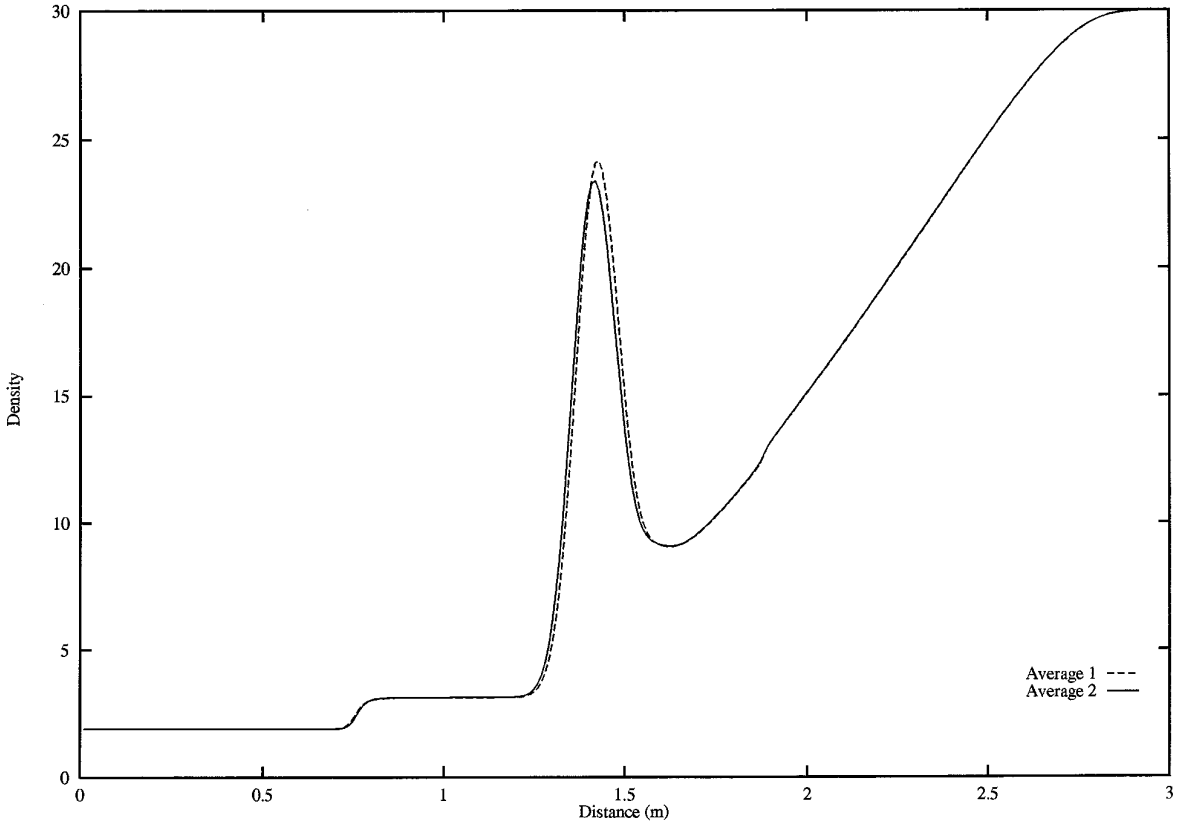
the nonconservative system (2.20) and the conservative system (2.23) have the same shock waves. Figure 3 shows the numerical results for two void fraction averages, the first one given by the expression (5.6) and the second one defined by $\tilde{\alpha}_l = \sqrt{\alpha_l^L \alpha_l^R}$. We remark that the results are slightly different.

PROBLEM 3 (Water faucet problem). This test, proposed by Ransom [13] consists in a vertical water jet, contained within a cylindrical channel, that is accelerated under the action of gravity. At the initial state, the pipe is filled with a uniform column of water surrounded by stagnant vapor, such that the void fraction is 0.2 and the column has a uniform pressure of 10^5Pa.

The boundary conditions are specified velocities of 10 m/s for the liquid and 0 m/s for the vapor at the inlet, and constant pressure at the outlet. The water faucet problem has a particularly simple analytical solution when pressure variation in the vapor phase is ignored [7]. This analytical solution was used as a code test in [14].

As illustrated on Fig. 4 a void wave develops and is propagated at liquid velocity. A countercurrent wave, propagating at vapor velocity, develops another shock. Once the void wave exits the pipe, a steady void profile is established. The calculation was carried out until a steady state is reached with 100 nodes and a constant CFL number equal to 0.9. Figure 5 shows the vapor void fraction profile at various time. These results clearly demonstrate the ability of the numerical scheme to capture discontinuities.

In order to test the convergence and the stability character of the scheme, computations have been made using a discretization with 50 and 200 nodes, but constant CFL numbers equal 0.9. Figure 6 gives the void fraction profile



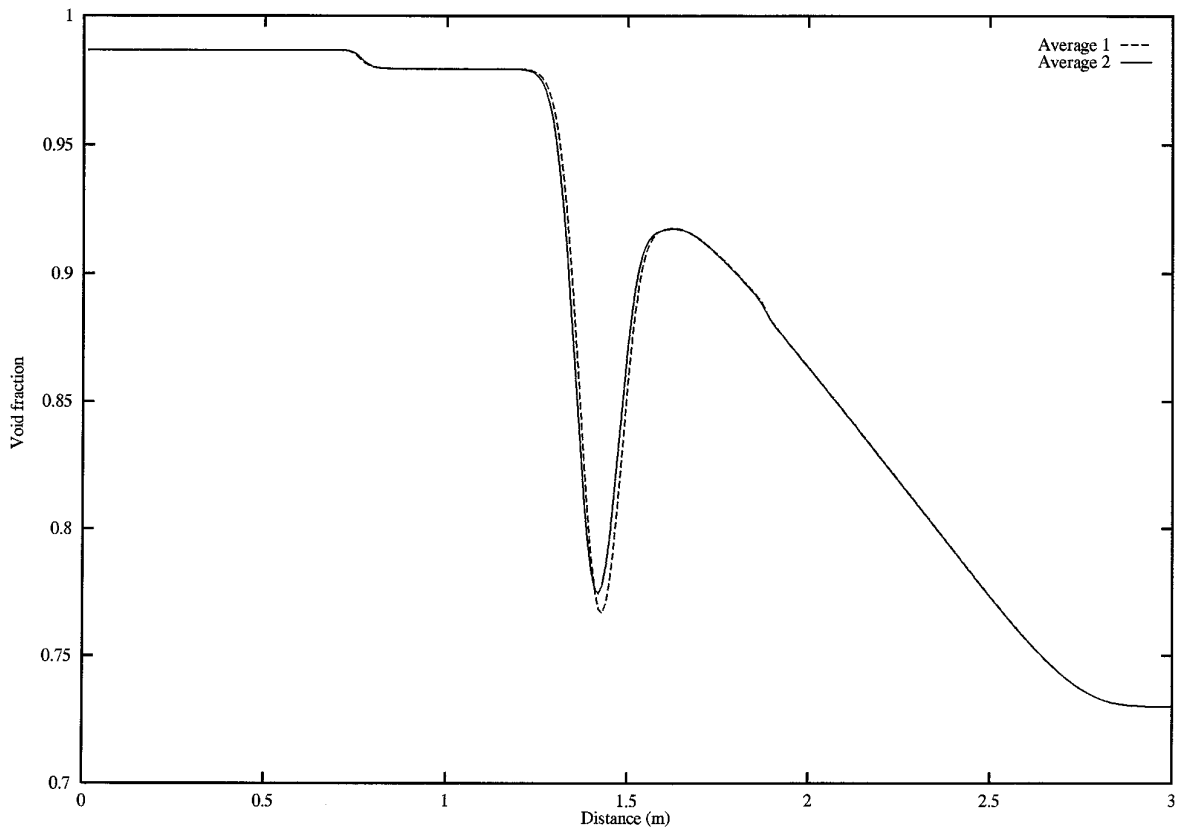


FIG. 3. Riemann problem for two void fraction averages.

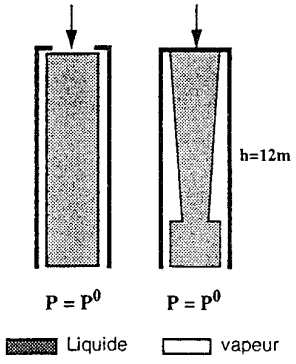


FIG. 4. Schematic of the water faucet problem.

for the various discretization. An interesting feature of the results shown in Fig. 6 is that there are no oscillations at the discontinuity of the void fraction.

6. CONCLUSION

We have presented here a numerical method for the simulation of an isentropic two-fluid model based on a linearized Riemann solver. A linearization of nonconservative product and an extension of Roe's scheme have been used in order to construct this approximate Riemann solver.

The solution of Riemann problem is unknown for our nonconservative system. Thus, there is no exact solver in order to check validity of the numerical solutions.

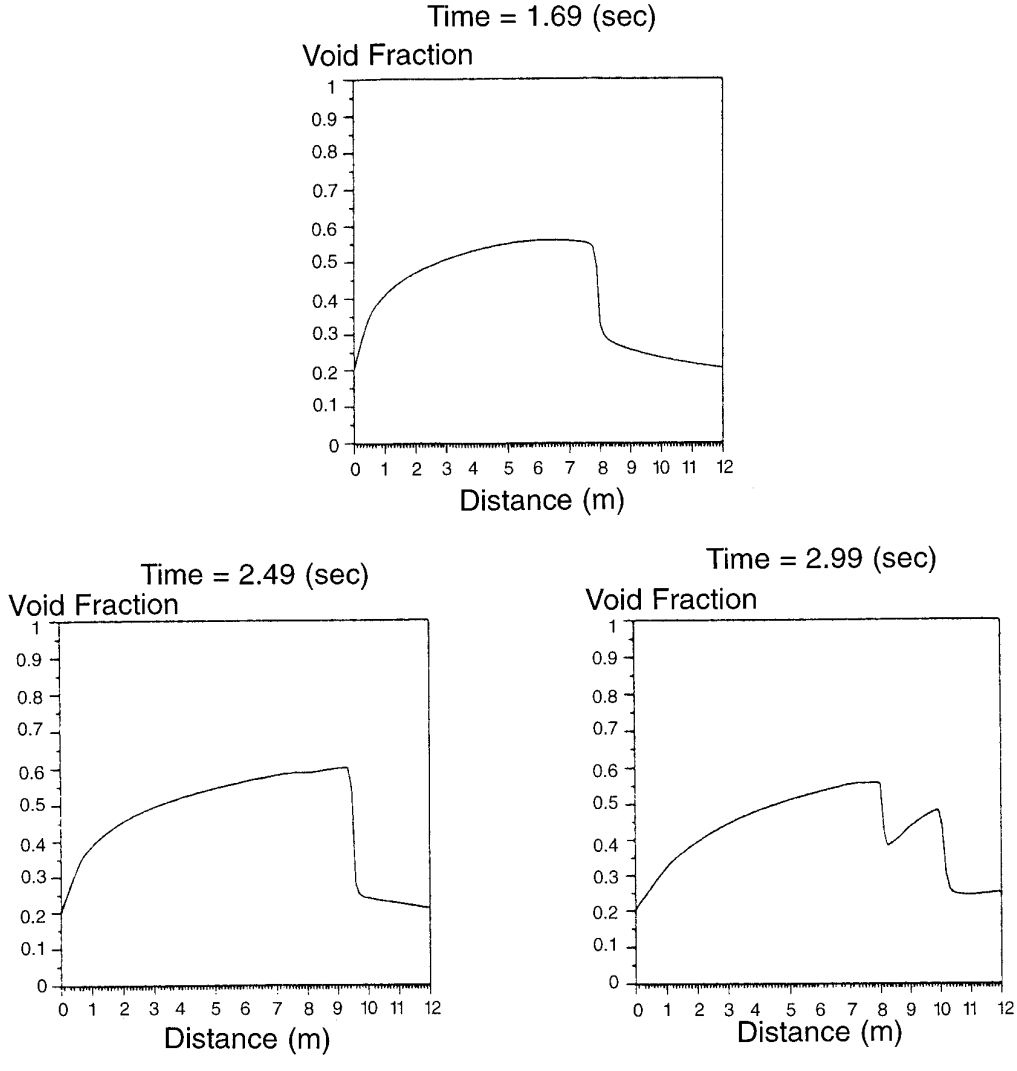


FIG. 5. Void fraction profile for the water faucet problem.

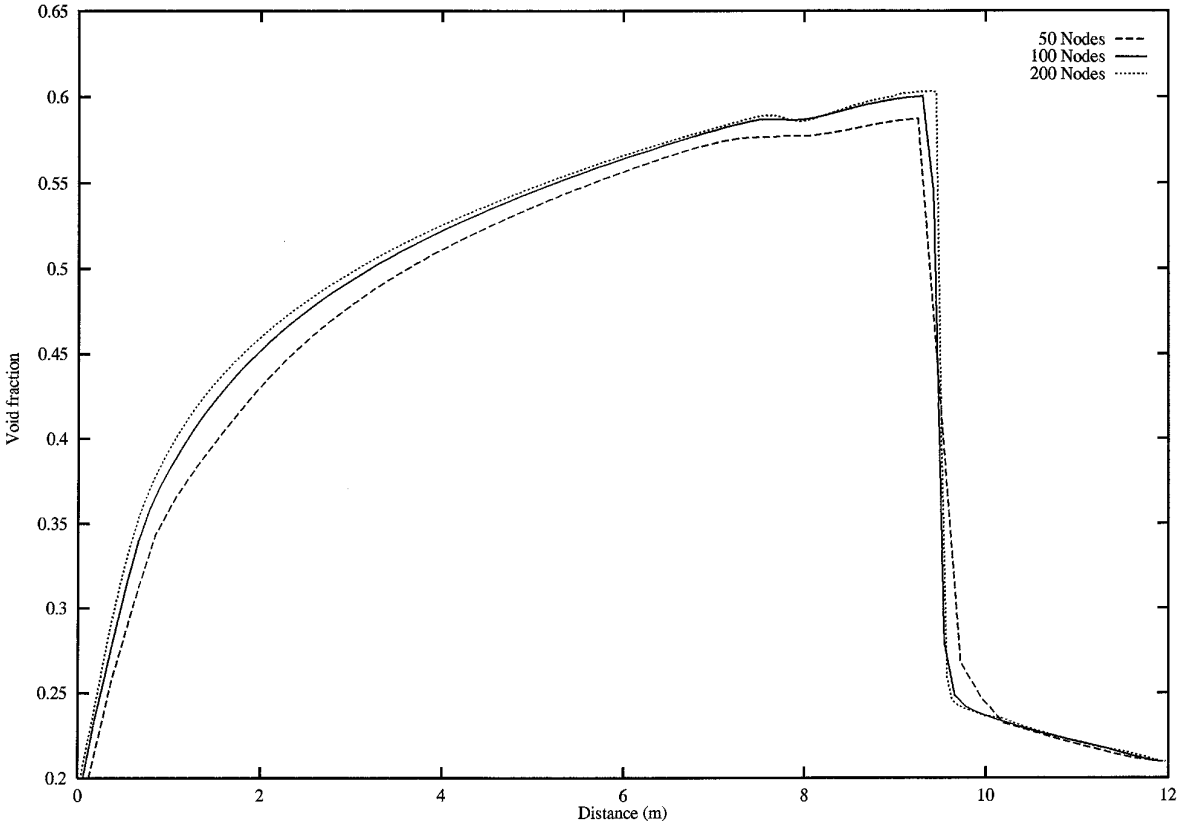


FIG. 6. Void fraction profile for the water faucet problem (*continued*).

Therefore, our results should be compared with the analytical solution for the physical problems, such as the water faucet problem. The comparison for this last case showed that the numerical method is stable and capable of generating accurate nonoscillating solutions for two-phase calculation. However, there is no theoretical background to ensure that computed shocks have always the correct speed.

This work is a first step to the study of a more complete two-fluid model consisting of mass, momentum, and energy balance equations for each phase. In fact, the difficulties in the computation of this model are, once more, the nonhyperbolic character of the governing equations and the nonconservative form of the system. Introducing an interface exchange term in the momentum equations is sufficient to make the complete model hyperbolic. Moreover, an equivalent conservative system can be found in order to linearize the nonconservative products.

REFERENCES

1. P. L. Roe, *J. Comput. Phys.* **43,** 357 (1981).
2. P. Glaister, *J. Comput. Phys.* **74,** 382 (1988).
3. H. C. Yee, NASA TM-89964, 1987 (unpublished).
4. I. Toumi, *J. Comput. Phys.* **102,** 360 (1992).
5. H. B. Stewart and B. Wendroff, *J. Comput. Phys.* **56,** 363 (1984).
6. D. R. Liles and W. H. Reed, *J. Comput. Phys.* **26,** 77 (1978).
7. J. A. Trapp and R. A. Riemke, *J. Comput. Phys.* **66,** 62 (1986).
8. M. Ishii, *Thermo-Fluid Dynamic Theory of Two-Phase Flow* (Eyrolles, Paris, 1975).
9. NREG/CR-3858, L.A-10157-MS, 1986 (unpublished).
10. J. C. Micaelli, 87-58, CEA, France, 1987 (unpublished).
11. V. H. Ransom and D. L. Hicks, *J. Comput. Phys.* **75,** 498 (1988).
12. V. H. Ranson *et al.,* NUREG/CR-1826, 1982 (unpublished).
13. V. H. Ransom, *Numerical Benchmark Tests, Multiphase Science and Technology,* Vol. 3, edited by G. F. Hewitt, J. M. Delhaye, and N. Zuber (Hemisphere, Washington, DC/New York, 1987).
14. V. H. Ransom and V. Mousseau, "Convergence and Accuracy of the RELAP5 Two-Phase Flow Model," in *Proceedings, ANS International Topical Meeting on Advances in Mathematics, Computations, and Reactor Physics, Pittsburgh, Pennsylvania, 1991.*
15. D. A. Drew, *Int. J. Multiphase Flow* **5,** 233 (1979).
16. R. T. Lahey, *Int. J. Multiphase Flow* **6,** 281 (1980).
17. L. Sainsaulieu, Thèse de Doctorat, Ecole Polytechnique, 1991 (unpublished).
18. Ph. Le Floch, *Commun. Part. Diff. Equa.* **13,** 669 (1989).
19. J. F. Colombeau and A. Y. Le Roux, in *Non Linear hyperbolic Problems,* Lecture Notes in Mathematics, Vol. 1270 (Springer-Verlag, New York/Berlin, 1987), p. 103.
20. A. Kumbaro, Thèse de Doctorat, Université Paris XI, 1992 (unpublished).

21. P. S. Gough and F. J. Zwarst, *AAIA J.* **17** (No. 1), 17 (1979).
22. E. F. Toro, Lecture Notes in Physics, No. 351 (Springer-Verlag, New York/Berlin, 1989), p. 472.
23. S. K. Godunov, *Math Sb.* **32,** 271 (1959).
24. A. Harten, P. D. Lax, and B. Van Leer, *SIAM Rev.* **25,** 35 (1983).
25. N. Zuber, *Chem. Eng. Sci.* **19** (1964).
26. R. T. Lahey Jr., D. A. Cheng, D. A. Drew, and J. E. Flaherty, in *AICHE 71st Annual Meeting at Miami Beach, Florida, 1978.*
27. T. Watanabe, M. Hirano, F. Tanabe, and H. Kamo, *Nucl. Eng. Design* **120** (1990).